The cosmological constant problem in codimension-two brane models

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Abstract

We discuss the possibility of a dynamical solution to the cosmological constant problem in the context of six-dimensional Einstein-Maxwell theory. A definite answer requires an understanding of the full bulk cosmology in the early universe, in which the bulk has time-dependent size and shape. We comment on the special properties of codimension two as compared to higher codimensions.

The presence of extra dimensions provides a framework in which the cosmological constant problem can be viewed from a different perspective. The question is no longer why the full spacetime curvature is so small, but rather why the four-dimensional (4D) curvature contains only such a small part of the total higher dimensional curvature. It has been known for about twenty years [1, 2] that in a certain subclass of six-dimensional solutions, namely those with time-independent size and shape of internal space, the 4D cosmological constant Λ_4 is a free integration constant of the general solution. The question remains why a solution with small Λ_4 should be dynamically selected. Recently this problem was reconsidered in the context of codimension-two braneworlds with conical sigularities [3, 4, 5, 6], with a bulk stabilized by magnetic flux [7].

We may view the solutions with static internal space as candidates for asymptotic solutions for large time t. For the approach to these asymptotic solutions, however, internal space is not expected to be static. The evolution of the universe in this early period with time varying geometry will decide to which value of Λ_4 the late universe will converge. Two scenarios are conceivable: The dynamical approach to the final value of the four-dimensional curvature (i.e. of Λ_4) may occur very early in cosmology, with a fixed Λ_4 since. This resembles the inflationary universe, where the final zero (or very tiny) value of the three-dimensional curvature is selected at a very early stage. (For the Friedmann solutions the initial density is a free integration constant, and its value very close to the critical density is selected during the early inflationary epoch.)

For the second alternative, the adjustment to the final value of Λ_4 is still going on in present cosmology. This will lead to a dynamical dark energy or quintessence [8]. In such a scenario the asymptotic value of Λ_4 is typically zero, and in present cosmology dark energy is expected to contribute of the same order as matter. However, potential problems arise now with time varying coupling constants.

Particular exact dynamic solutions of the 6D Einstein-Maxwell theory have been recently found for both scenarios [9]. They are, however, special in the sense that only the size, not the geometric shape of internal space changes with time, and that there is no warping. A dynamical solution of the cosmological constant problem in early cosmology is not expected in such a restricted setting and actually not found. An investigation of this question requires a time varying geometry and warping within the most general class of solutions consistent with the symmetries. Recent progress towards an understanding of this complicated dynamics was made by Vinet and Cline [4]. They considered different types of singularities, allowing for a general equation of state on the brane. The authors find no self-tuning mechanism. However, the results of ref. [4] are by far not sufficient for a definite answer. The limitation of their work is the restriction to small perturbations around an essentially static bulk: the "football shaped" solution [5]. If the self-tuning of the 4D cosmological constant takes place in the very early universe, there is no reason to assume that the bulk was static, or even almost static, at that time. A general analysis should take place in a bulk with the most general (non-perturbatively) time-dependent size and shape which is consistent with the symmetries. The curvature may then be dynamically shifted between the 4D part, the 2D part and the warping, implying an effective time-dependent 4D cosmological constant.

This can happen even if there are no time-dependent sources at all, only the 6D cosmological constant and a constant magnetic flux, as we demonstrated in ref.[9]: The quintessence field ϕ of the effective 4D theory is given by the radius of the internal space and is therefore related to its curvature. On the other hand, the potential $V(\phi)$ and the time derivatives of ϕ induce some curvature in the 4D world. The dynamics of ϕ is nothing else but an effective description of the interaction between the different parts (2D and 4D) of the total six-dimensional curvature.

The allowance for a more general geometry complicates our subject immensely. The "football" shaped cosmology was characterized by a few constants (the brane tension and the monopole number) and one single function $\phi(t)$. Within the more general geometry, there is an infinite number of degrees of freedom, corresponding in the effective four-dimensional world to an infinite set of scalar fields in the singlet representation of the symmetry group. Below we derive the most general ansatz for the six-dimensional metric with the symmetries of three-dimensional rotations and translations and internal U(1) isometry, and we have also computed the corresponding field equations.

For an investigation of cosmological solutions we aim to answer the question: Is there a large class of initial conditions for which the bulk comes to rest asymptotically leaving a very small or zero 4D curvature? To study this question, we adopt a bulk-based point of view [10], in which the singularities (branes) are seen as properties of the bulk geometry (such as the mass of a Schwarzschild black hole may be seen as an integration

constant of the vacuum geometry). This simplifies our task since only the field equations in the bulk need to be solved. The singularities will certainly play an important role in the development of the bulk. It is of particular interest how bulk fields respond to the singularities, and if a part of the energy of the fields may "fall" into them (like matter falls into a black hole). Due to the high complexity of the field equations we have not yet achieved to answer these questions in the present paper. The aim of the present note is therefore more modest, i.e. to clarify the problem for general bulk geometries and develop a strategy for further investigations. In the text we mainly refer to the Kaluza-Klein context which implies that the two internal dimensions are compactified on a scale not much different from the Planck scale. Nevertheless, most of our results remain valid in braneworld scenarios with large extra dimensions.

As our first general observation we note that codimension two is a very special case, for several reasons on which we comment below. We already mentioned the speciality of codimension one in an earlier paper [10], in particular the fact that the position of a codimension-one brane cannot be detected by a "test particle" in the bulk. In contrast, for codimension two or higher the type and strength of the singularity can be inferred from the properties of the bulk geometry. Still codimension two is special since there exists a type of brane that is not possible for any higher codimension: the deficit angle brane. Using coordinates x^μ for the four large dimensions and ρ and θ (with $0 \leq \theta < 2\pi$) as cylindric coordinates for the internal space, such a brane, or conical singularity, can be described in the following way: The metric components $g_{\mu\nu}$ have well defined, finite values at the position of such a singularity (at $\rho = 0$, say), while $g_{\theta\theta}$ is proportional to ρ^2 in the vicinity of the brane. This is the usual behavior of cylindric coordinates with radial coordinate ρ. The only effect of the singularity is a deficit angle Δ, which is expressed in the metric by the fact that

$$g_{\theta\theta} = \left(1 - \frac{\Delta}{2\pi}\right)\rho^2 + O(\rho^3). \tag{1}$$

The infinite curvature at $\rho = 0$ is not "visible" from outside, by which we mean that the curvature R (and in fact any invariant formed from the Riemann tensor) remains finite in the limit $\rho \to 0$. The curvature and the corresponding brane tension are of the delta function type. The finiteness of the curvature implies that there are no attractive forces towards the brane, at least none with a divergent behavior.

Such a type of singularity exists only in the codimension two case. Otherwise all singularities are not of the delta function type, i.e. they are locally "visible" from outside by Riemann tensor components that diverge as $\rho \to 0$, and hence diverging forces as one approaches them. The reason for the existence of deficit angle branes in codimension two is that one may "cut out" a part (i.e. the deficit angle) of the circle described by the θ coordinate at constant ρ without inducing any curvature on it (since it is a one-dimensional object). For $D > 2$, the sphere S^{D-1} described by coordinates θ_α at constant ρ does have curvature, and "cutting out" some part of it does not work. Or, equivalently, multiplying the $g_{\theta\theta}$'s by a constant factor induces a change in the curvature which diverges as one approaches $\rho = 0$.

In fact, the deficit angle brane is a special type of a Kasner singularity. Consider, for simplicity, a static vacuum singularity at $\rho = 0$, i.e. all metric components are functions of ρ only. We may then normalize $g_{\rho\rho}$ to 1, and the most general metric (in 4+D dimensions) consistent with our symmetries (in particular internal SO(D) isometry) is

$$ds^2 = -c^2(\rho)dt^2 + a^2(\rho)(dx^i)^2 + b^2(\rho)\tilde{g}_{\alpha\beta}(\theta)d\theta^\alpha d\theta^\beta + d\rho^2. \tag{2}$$

In the vicinity of the singularity, the vacuum Einstein equations admit solutions of the form

$$c \sim \rho^{p_1}, \quad a \sim \rho^{p_2}, \quad b \sim \rho^{p_3} \tag{3}$$

with

$$p_1 + 3p_2 + (D-1)p_3 = p_1^2 + 3p_2^2 + (D-1)p_3^2 = 1. \tag{4}$$

The deficit angle brane corresponds to the very special solution with $p_1 = p_2 = 0$ and $p_3 = 1$ which exists only for $D = 2$. In all other cases the $g_{\mu\nu}$ components become irregular at $\rho = 0$ (either zero or infinite), and some components of the Riemann tensor diverge.

In the presence of bulk matter, the singularities may have an important influence on the cosmological evolution. Except for the deficit angle case, they may attract the matter and force it to fall into them, making the singularities grow (as it is familiar for black holes). As an example, consider $p_2 = 0$ (i.e. constant a) where

$$p_1 = \frac{1 \pm \sqrt{1 + D(D-2)}}{D}, \quad p_3 = \frac{1}{D} \mp \frac{\sqrt{1 + D(D-2)}}{D(D-1)}. \tag{5}$$

For $D = 3$ one has a solution with a "black hole singularity" ($p_1 = -\frac{1}{3}$, $p_3 = \frac{2}{3}$) in internal space. One would expect the existence of solutions where matter falls into this singularity, thereby changing the strength of the singularity or the associated brane tension (given by the "mass" of the black hole). Of course, the analogy of such a cosmological solution with a black hole is only formal. In the effective four-dimensional theory there is no local object since the solution is actually a direct product of time-warped internal space and flat three-dimensional geometry. The time singularity appears only for a particular point in internal space. Integrating over internal space may lead to a perfectly regular time in the effective four-dimensional world.

We may take this discussion as a warning that results for singularities with codimension two should not be too easily generalized to higher codimension. It is well conceivable that the strength of singularities does not change with time for codimension two branes whereas it generically does for higher codimension.

A second speciality of codimension two arises when one considers the most general metric consistent with certain symmetries: We want to look for cosmological solutions with a general shape of the two-dimensional internal space. (Static and de Sitter-like solutions were described in refs. [7, 1, 2].) The first step of a dynamical investigation is the selection of an appropriate ansatz for the metric. We will see that the determination of the most general metric consistent with the symmetries is nontrivial and actually extends

beyond the metrics considered so far [4]. We require the following symmetries: three-dimensional translation and rotation invariance, acting on the coordinates x^i, and a U(1) symmetry, acting on the coordinate $\theta \in [0, 2\pi]$. No metric function should depend on x^i or θ, and no direction in the three-dimensional space should be preferred. (For simplicity, we will take this space to be flat, so that the metric components g_{ij} are $a^2(t,\rho)\delta_{ij}$.) We have to find the most general metric consistent with these symmetries.

Isotropy forbids metric components g_{ti}, $g_{\rho i}$ and $g_{\theta i}$, since these would select preferred directions in three-space, e.g. by the three-vector (g_{t1}, g_{t2}, g_{t3}). The other off-diagonal metric components $g_{t\rho}$, $g_{t\theta}$ and $g_{\rho\theta}$ are allowed, as long as they are functions of t and ρ only. Up to now we have identified the most general metric consistent with the symmetries as

$$\begin{aligned} ds^2 = -c^2(t,\rho)dt^2 &+ a^2(t,\rho)dx^i dx^i + b^2(t,\rho)d\theta^2 + n^2(t,\rho)d\rho^2 \\ &+ 2w(t,\rho)dtd\rho + 2u(t,\rho)dtd\theta + 2v(t,\rho)d\rho d\theta. \end{aligned} \tag{6}$$

The next step is to look how far this line element can be simplified by a coordinate transformation. Therefore one has to find the possible transformations consistent with the symmetries, which should still be represented by the new coordinates $x^{i'}$ and θ'. Transformations can never depend on θ, since this would lead to metric functions depending on θ'; for example $t \to t' = t + \delta t(\theta)$, $\theta \to \theta' = \theta$ would imply $t = t' - \delta t(\theta')$, and so $f(t) \to f'(t', \theta')$ for any function f. Similarly, t', θ' and ρ' cannot depend on x^i. Furthermore, for $\theta \to \theta'$, one has

$$g^{\theta'\theta'} = \left(\frac{\partial\theta'}{\partial\theta}\right)^2 g^{\theta\theta}, \tag{7}$$

and we impose $\partial\theta'/\partial\theta = 1$, since θ' should be in the interval $[0, 2\pi]$.

Transformations of x^i cannot depend on t or ρ, since this would lead to forbidden components via

$$g^{t'i'} = \frac{\partial t'}{\partial t}\frac{\partial x^{i'}}{\partial t} g^{tt}, \tag{8}$$

and similarly for $g^{\rho i}$. Obviously, the only effect of a transformation $x^i \to x^{i'}(x^j)$ could be a rescaling of three-dimensional space, so we can forget about them in this context. We are left with the following possibilities:

$$\begin{aligned} x^i &\to x^i, \qquad \theta \to \theta + \delta(t,\rho), \\ t &\to t'(t,\rho), \qquad \rho \to \rho'(t,\rho). \end{aligned} \tag{9}$$

There are three off-diagonal metric components, $g_{t\rho}$, $g_{t\theta}$ and $g_{\rho\theta}$, and one might think that these can be removed by the three remaining coordinate transformations. It turns out that this is in general not true. The reason for that is essentially the U(1) symmetry. (In fact, the metric *can* always be diagonalized, but then in general the new coordinate θ' will not reflect the U(1) symmetry any more, and fields will depend on θ'.) To see this, consider the inverse of the metric. The components $g_{t\theta}$ and $g_{\rho\theta}$ will be zero if and only if

$g^{t\theta}$ and $g^{\rho\theta}$ are zero. The condition that this happens after a coordinate transformation of the type (9) is

$$g^{t'\theta'} = \frac{\partial t'}{\partial t}\left(g^{t\theta} + \frac{\partial \theta'}{\partial t}g^{tt} + \frac{\partial \theta'}{\partial \rho}g^{t\rho}\right) + \frac{\partial t'}{\partial \rho}\left(g^{\rho\theta} + \frac{\partial \theta'}{\partial t}g^{\rho t} + \frac{\partial \theta'}{\partial \rho}g^{\rho\rho}\right) = 0, \tag{10}$$

$$g^{\rho'\theta'} = \frac{\partial \rho'}{\partial t}\left(g^{t\theta} + \frac{\partial \theta'}{\partial t}g^{tt} + \frac{\partial \theta'}{\partial \rho}g^{t\rho}\right) + \frac{\partial \rho'}{\partial \rho}\left(g^{\rho\theta} + \frac{\partial \theta'}{\partial t}g^{\rho t} + \frac{\partial \theta'}{\partial \rho}g^{\rho\rho}\right) = 0. \tag{11}$$

A solution of these differential equations implies either that the Jacobi determinant of the (ρ, t) transformation vanishes,

$$det\begin{pmatrix} \frac{\partial t'}{\partial t} & \frac{\partial t'}{\partial \rho} \\ \frac{\partial \rho'}{\partial t} & \frac{\partial \rho'}{\partial \rho} \end{pmatrix} = 0, \tag{12}$$

which is not possible, or that the brackets vanish. But the second possibility consists of two conditions for the function θ', which can in general not be fulfilled simultaneously.

One concludes that generally only one of the two components $g^{t\theta}$ and $g^{\rho\theta}$ can be set to zero (in contrast to [4]). A procedure to simplify the metric (6) could look as follows: Use the freedom for t' and ρ' to annihilate $g^{t\rho}$ and for one further simplification, e.g. to arrange that ${g_{tt}}' = -{g_{ii}}'$, i.e. to make time conformal with respect to space. Then use the freedom for θ' to annihilate either $g^{t\theta}$ or $g^{\rho\theta}$. The simplified line element is then

$$ds^2 = a^2(t,\rho)(-dt^2 + dx^i dx^i) + b^2(t,\rho)d\theta^2 + n^2(t,\rho)d\rho^2 + 2u(t,\rho)dtd\theta, \tag{13}$$

or similarly with $2v(t,\rho)d\rho d\theta$ instead of $2u(t,\rho)dtd\theta$. In the effective four-dimensional picture u corresponds to the time component of an abelian gauge field (hence some kind of electric potential), since $g_{\theta\mu}$ integrated over internal space is the gauge field corresponding to the U(1) isometry. On the other hand v corresponds to a scalar field. The fact that a degree of freedom can be shifted between a scalar field and the component of a gauge field is a familiar fact in particle physics.

The presence of off-diagonal metric components $g_{t\theta}$, $g_{\rho\theta}$, which cannot be transformed away simultaneously, is a special feature of codimension-two models. Consider $D > 2$ internal dimensions, and $D-1$ of these dimensions, represented by coordinates θ_α, were symmetric under, say, SO(D), then the $g_{t\theta}$ and $g_{\rho\theta}$ components would be forbidden, because they would select preferred directions in the $D-1$-dimensional space, in conflict with the SO(D) symmetry. The difference is that a U(1) "rotation" is a translation rather than a rotation. In this sense a codimension-two spacetime is more complicated than a higher-dimensional one.

A third special feature of codimension two is that internal space can be compactified and stabilized by a gauge field A_B in a monopole configuration (capital indices run over all six dimensions). Six-dimensional Einstein-Maxwell theory [7] according to the action

$$S = \int d^6x\sqrt{-g}\left\{-\frac{M_6^4}{2}R + \lambda_6 + \frac{1}{4}F^{AB}F_{AB}\right\}, \tag{14}$$

is a convenient toy model for higher dimensional scenarios. Here M_6 is the reduced Planck mass corresponding to six-dimensional gravity, λ_6 is a cosmological constant term and F_{AB} is the field tensor of the gauge field. Including the gauge field into our considerations, we find that the three components A_t, A_ρ and A_θ are allowed by the symmetries. One can choose to set either A_t or A_ρ to zero by a gauge transformation. This is similar to the choice between $g^{t\theta}$ and $g^{\rho\theta}$ described above.

Comparing this cosmological Einstein-Maxwell system to the static case, one finds that the ordinary differential equations (containing only ρ-derivatives) are generalized to partial differential equations, containing t- and ρ-derivatives. The three functions a, b and A_θ, which are already present in the static case, are accompanied by three more functions: n, u or v, and A_t or A_ρ.

We have computed the field equations for the six independent functions of t and ρ. As an example we give the $(tt)-$ component of Einstein's equations, in the gauge $a = c$ and $w = v = 0$:

$$\begin{aligned} G^t_t \equiv & -\frac{1}{a^2(1+q^2)}\left(3\frac{\dot{a}^2}{a^2} + 3\frac{\dot{a}\dot{b}}{ab} + 3\frac{\dot{a}\dot{n}}{an} + \frac{\dot{b}\dot{n}}{bn}\right) + \frac{1}{n^2}\left(3\frac{a'^2}{a^2} - 3\frac{a'n'}{an} + 3\frac{a''}{a}\right) \\ & + \frac{1}{n^2(1+q^2)}\left(3\frac{a'b'}{ab} - \frac{b'n'}{bn} - \frac{n'u'q^2}{2nu} + \frac{b''}{b} + \frac{u''q^2}{2u}\right) \\ & + \frac{q^2}{n^2(1+q^2)^2}\left(\frac{a'b'}{ab} + \frac{b'^2}{b^2} + \frac{a'u'}{au}(1+\frac{3}{2}q^2) - \frac{3b'u'q^2}{2bu} + \frac{u'^2}{4u^2}(1-q^2)\right) \\ & = 8\pi G_6 T^t_t \equiv \frac{4\pi G_6}{1+q^2}\left(-\frac{\dot{A}_\theta^2}{a^2b^2} - \frac{A_t'^2}{a^2n^2} - \frac{A_\theta'^2}{n^2b^2}\right). \end{aligned} \tag{15}$$

Here dots and primes denote derivatives with respect to t and ρ, respectively, and we use the abbreviation $q^2 \equiv u^2/(a^2b^2)$. The equations for the other components are of similar length and are not displayed here. A full numerical analysis of this system would involve as initial conditions twelve functions of ρ (four metric and two gauge field components and their first time derivatives at some initial time t_0) which are subject to three constraint equations, namely the $(tt)-$, $(t\rho)-$ and $(t\theta)-$ components of Einstein's equations, which contain no second time derivatives. The time evolution is determined by the $(ii)-$, $(\theta\theta)-$, $(\rho\rho)-$ and $(\theta\rho)$- components of Einstein's equations and two equations for the gauge field.

Again we want to compare this to an Einstein-Maxwell system in higher codimensions. We already showed that the metric components $g_{t\theta_\alpha}$ and $g_{\rho\theta_\alpha}$ are not consistent with a symmetry larger than U(1) acting on the θ coordinates. For the gauge field the situation is slightly different. For specific solutions (solitons) a component A_{θ_α} may be allowed even if the internal symmetry is larger than U(1). An example is the monopole solution on S^2. Although $A_\rho = 0$ and $A_\theta \neq 0$, the θ-direction is not preferred physically. A coordinate transformation may be accompanied by a gauge transformation, so that the transformed A-field lies in the new θ-direction. An analogous procedure does not work for the metric tensor, since the gauge transformations are the coordinate transformations themselves. But the components A_ρ and A_t are not necessary in the $D > 2$ case: Without the $g_{t\theta}$ and $g_{\rho\theta}$ metric components, the components ${G^t}_\theta$ and ${G^\rho}_\theta$ of the Einstein tensor are identically zero. The corresponding components of the energy momentum tensor induced by the

Maxwell field,

$$T_{AB}^{(F)} = F_{AC}F_B{}^C - \frac{1}{4}F_{CD}F^{CD}g_{AB}, \tag{16}$$

are then also zero, which implies $F_{\rho t} = 0$ (as long as A_{θ_α} is non-trivial) and so A_ρ and A_t are pure gauge. Compared to the codimension-two case, the higher codimensions therefore involve two functions less: Only a, b, n and A_θ remain after appropriate simplifications.

We already mentioned that, after removing two of the off-diagonal metric components, there remains still one degree of freedom for coordinate changes in order to bring the metric into a pleasant form. We chose to use this freedom to make time conformal, i.e. $-g_{tt} = g_{ii}$. We warn, however, that this may often not be the convenient choice, because the corresponding time coordinate may be different from the "physical" time. To explain this, remember that in usual four-dimensional cosmology, time can be made conformal by a transformation $t \to \tau(t)$. In the six-dimensional model, we need instead transformations $t, \rho \to t'(t,\rho), \rho'(t,\rho)$ in order to bring the metric into the required form. This may mix the time and ρ coordinate to some extent. The effective four-dimensional Lagrangian is obtained by integrating out internal space in the form

$$L_{eff}(t, x^i) \sim \int d\rho d\theta \sqrt{g_{int}} L(t, x^i, \rho, \theta), \tag{17}$$

or similar, where g_{int} is the determinant of the metric of the internal space. This effective Lagrangian obviously depends on the choice of the (t,ρ) frame one uses. Nevertheless, all the effective Lagrangians derived from different choices of t and ρ must describe the same physics because they are obtained from the same six-dimensional theory. Such an equivalence between different Lagrangians will in general not be seen easily, because there is an infinite number of fields mixed with each other when going from one frame to the other.

To illustrate this, we consider the example of five-dimensional Kaluza-Klein theory with action

$$S = -\frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\, R. \tag{18}$$

The four large dimensions are again parametrized by coordinates x^μ, and the fifth coordinate y runs from 0 to $2\pi r$. If one writes the metric in the form

$$g_{AB} = \phi^{-1/3}\begin{pmatrix} \tilde{g}_{\mu\nu}A_\mu A_\nu \phi & A_\nu\phi \\ A_\mu\phi & \phi \end{pmatrix}, \tag{19}$$

integration over y leads to the following 4D action for the zero modes:

$$S_{eff}^{(0)} = \frac{1}{16\pi G_4}\int d^4x\sqrt{-\tilde{g}}\left(-\tilde{R}^{(0)} + \frac{1}{4}\phi^{(0)}F_{\mu\nu}^{(0)}F^{\mu\nu(0)} - \frac{1}{6}\frac{\partial_\mu\phi^{(0)}\partial^\mu\phi^{(0)}}{(\phi^{(0)})^2}\right). \tag{20}$$

Here $G_4 = G_5/2\pi r$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the superscript (0) denotes that in the Fourier expansion of components with respect to y only the zero modes are taken into account, and a tilde denotes a quantity constructed from $\tilde{g}_{\mu\nu}$.

Now we perform a small coordinate transformation, affecting only the time coordinate in the form $t \to t' = t - \epsilon \sin \frac{y}{r}$. The corresponding change of the metric is (to order ϵ)

$$g'_{ty}(x^i, t', y) = g_{ty}(x^i, t(t', y), y) + \epsilon \, \frac{g_{tt}(x^i, t(t', y), y)}{r} \cos \frac{y}{r}, \tag{21}$$

$$g'_{yy}(x^i, t', y) = g_{yy}(x^i, t(t', y), y) + 2\epsilon \, \frac{g_{ty}(x^i, t(t', y), y)}{r} \cos \frac{y}{r}. \tag{22}$$

This transformation does not only mix the fields ϕ, A_μ and $\tilde{g}_{\mu\nu}$ in a complicated way. It also changes the line of y-integration defined by $x^\mu = const$. In the transformation of the fields this is reflected by additional time derivatives, e.g.

$$\phi' = \phi \, \left(1 + 3\epsilon \, \frac{A_t}{r} \cos \frac{y}{r} + \epsilon \, \frac{\dot{\phi}}{\phi} \sin \frac{y}{r} \right). \tag{23}$$

The 4D action for the zero modes of the new fields will again be eq.(20), but these zero modes are not only combinations of the old zero modes. They contain admixtures of higher Fourier modes of the original fields (due to the $\cos \frac{y}{r}$ term) and even of their time derivatives.

For a given solution of the field equations, there are certainly choices of coordinates in which the four-dimensional world looks simpler than in others. In some situations there may be a clear and unique preferred frame which identifies a "physical" time coordinate. The "physical" time is easily identified when there is a timelike Killing vector. Returning to the six-dimensional model, such a Killing vector is given for the Kasner solutions mentioned before. In a situation like the static football shaped solution ($p_1 = p_2 = 0$ at the singularities), where time and three-space are not differently warped, an appropriate frame has necessarily $-g_{tt} = g_{ii}$. For Kasner solutions with $p_1 \neq p_2$, like the aforementioned black hole type singularities, time and three-space are warped differently, and the physical time (with time axis parallel to the Killing vector) corresponds to a frame with $-g_{tt} \neq g_{ii}$. In this frame all metric components depend only on ρ (cf eq (2)). In general, Lorentz invariance will be broken in the effective 4D world corresponding to such a solution. By suitable (t, ρ) transformations one still finds local charts with $-g_{t't'} = g_{ii}$, but these coordinates do not represent the symmetry of the solution, since now functions depend on t' and ρ': $f(\rho) = f(\rho(t', \rho'))$. Everything would look unnecessarily complicated in such a frame, which is therefore "unphysical".

Even for late cosmology, say the present epoch, there is no exact timelike Killing vector, only an approximate one. This means that for some choice of coordinate frame variables vary only very slowly with time. The identification of the "correct" time is more complicated in such a situation than in those with exact Killing vector. For a wrong choice of (ρ, t)-frame variables may vary much too fast with time and the geometry seems to get totally distorted. This is similar to (but worse than) the unphysical gauge modes appearing in some approaches of cosmological perturbation theory (e.g. in synchronous gauge). The identification of the time coordinate relevant for the effective four-dimensional physics is a serious task in higher-dimensional cosmology. Conclusions based on the "conformal gauge" ($a = c$ in eq.(6)) can easily be misleading. In practice, this means that it may be

advisable to work with a metric that is even more general than the ansatz (13).

In summary, we have computed the field equations for the six-dimensional Einstein-Maxwell theory for the most general ansatz of the metric and gauge fields consistent with the symmetries of three dimensional rotations and translations and a U(1)-isometry. A crucial issue for a possible dynamical solution of the cosmological constant problem is the possibility that the brane tension changes with time. For a restricted ansatz it was found that this does not happen for this system [3], but we would like to emphasize that a complete answer needs the most general ansatz for the metric. Therefore a dynamical solution to the cosmological constant problem in the context of six-dimensional brane or Kaluza-Klein models is so far not ruled out, not even in the case of infinitely thin deficit angle branes. An answer to the question requires a much more detailed understanding of the early universe dynamics with a time-dependent bulk geometry. As we have shown, such an understanding is complicated by the fact that the four-dimensional interpretation of the six-dimensional dynamics is far from clear in the absence of a timelike Killing vector.

Furthermore, we have pointed out that codimension two is a very special case for several reasons:
(i) There exist conical singularities (deficit angle branes) which do not induce any attractive forces in the bulk. (ii) The metric is relatively complicated even if one requires that all quantities depend only on one internal coordinate ρ and on time. (iii) A Maxwell field can compactify and stabilize the internal space.

We have argued by analogy with the dynamical black hole geometries with infalling matter in the four-dimensional world that in general the strength of a singularity can vary with time. Once such time varying "brane tensions" can be desribed properly in a higher-dimensional world the issue of the cosmological constant or quintessence may show new, unexpected facets.

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